A Study of Fishery Resource With Reserve and Unreserved Area

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Abstract – In this paper we have formulated and analyzed two preys one predator model with one prey dispersal in a two patch environment, one accessible to both prey and predators (unreserved rea) and the other one being a refuge for the prey (reserved area). The prey refuge (reserved area) constitutes a reserve zone of prey and fishing is not permitted, while the unreserved zone area is an open-access fishery zone. The existence of possible steady states, along with their local and global stability is discussed.

Keywords – Prey , Predator, Unreserved Area, Global Stability.

1. INTRODUCTION

It is well known that many species have already become extinct and many others are at the verge of extinction due to several natural or manmade reasons like over exploitation, indiscriminate harvesting, over predation, environmental pollution, loss of habitat and mismanagement of natural resources etc. Over the past decades, mathematics has made a considerable impact as a tool to model and understand biological phenomena. Freedman[6] described the single species migration in two habitat with persistence and extinction. Collings[8] studied the nonlinear behavior of predator-prey model with refuge protecting a constant proportion of prey and wit temperature dependent parameters chosen appropriately for a mite interaction on fruit species. Shukla et.al.[9] considered effect of changing habitat on survival of species. Zhang et.al.[17] presented in this paper establish a mathematical model of two species with stage structured and the relation of predator prey. Dubey et.al.[2]describe the model for fishery resource with reserve area. Kar [13] studied on the effect of harvesting on population growth with time delay.

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Kar and Chaudhuri [16], investigated a dynamic reaction model in the case of a prey-predator type fishery system, where only the prey species is subjected to harvesting, taking taxation as a control instrument. Dubey et.al. [3] , proposed and analyzed a mathematical model to study the dynamics of one prey, two predators system with ratio dependent predators growth rate. Kar and Matsuda [15] investigated a prey-predator model with Holling type of predation and harvesting of nature predator species. Dubey[1] analyzed a dynamic model for coexistence and stability behavior of predator-prey system in the habitat. Braza[12] analyzed a two predator, one prey model in which one predator interferes significantly with other. Kar. et. al. [14], in their paper, offer some mathematical analysis of the dynamics of a two prey, one predator system in the presence of a time delay. Singh et.al. [4] proposed a generalized mathematical model to study the depletion of resources by two kinds of populations, one is weaker and others stronger. A model of prey-predator with a generalized transmission function for unsaturated zone has been analyzed by Mehta et al. [7]. Recently Agrawal [11] developed a simple two species prey predator model in which the prey dispersal in a two patch environment.

We study a prey-predator system in a two patch environment: one accessible to both prey and predators (patch 1) and the other one being a refuge for the prey (patch 2). Each patch is supposed to be homogenous. We suppose that the prey migrate between the two patches randomly.

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The growth of prey in each patch in the absence of predators is assumed to be logistic.

2. Mathematical Model

Mathematical model of ecological system, reflecting these problems, has been given in B. Dubey [1] . The paper is concerned with the following prey predator system

$$
\begin{aligned}
&\frac{dx}{dt} = rx(1-\frac{x}{Ax+K}) - \sigma_1 x + \sigma_2 y - \beta_1 xz,\\
&\frac{dy}{dt} = sy(1-\frac{y}{L}) + \sigma_1 x - \sigma_2 y, \\
&\frac{dz}{dt} = Q(z) - \beta_0 z,\\
&x(0) \ge 0, y(0) \ge 0, z(0) \ge 0.
\end{aligned}
\tag{1}
$$

Let x(t) the density of prey species in unreserved zone, y(t) the density of prey species in reserved zone and z(t) the density of the predator species at any time t ≥ 0. Where no fishing is permitted at time t. Let σ_1 be the migration rate coefficient of prey species from unreserved to reserved zone and σ_2 the migration rate coefficient of prey species from reserved to unreserved zone. It is assumed that the prey species in both zones are growing logistically. Here r and s are intrinsic growth rate coefficient of the prey species with K and L are carrying capacities in unreserved and reserved zones respectively. β_1 is the depletion rate coefficient of the prey species due to the predator, and β_0 is the natural death rate coefficient of the predator species. A is the constant coefficient with prey. All the parameters are assumed to be positive.

In model (1), the function Q(z) represent the growth rate of predator. We consider in this model predator is partially dependent on the prey .So that

$$Q(z) = bz\left(1-\frac{z}{M_0}\right) + \beta_2 xz \tag{2}$$

The prey species of density x(t) can be thought of as an alternative resource for the predator. By denoting $a = b - \beta_0 > 0$, $M = M_0\,(b - \beta_0)\,/b$ we note that the equation (2) of model (1) can be written as

$$\frac{dz}{dt} = az(1-\frac{z}{M}) + \beta_2 xz \ . \tag{3}$$

3. Existence of Equilibria

When Q (z) satisfies equation (3) . Then it can be checked that model (1) has four nonnegative equilibria, namely, $F_0(0,0,0), F_1(0,0,M), F_2(\tilde{x},\tilde{y},0), F^*(x^*,y^*,z^*)$. The equilibriums F_0 and F_1 obviously exists and we shall show the existence of equilibrium $F_2(\tilde{x},\tilde{y},0)$ as follows:

Existence of $F_2(\tilde{x},\tilde{y},0)$

Further $\tilde{x}$ to be positive, we must have

$$\tilde{x} > \frac{K}{r}(r - \sigma_1). \tag{4}$$

To see the existence of F^*, we note that x^*, y^*, z^* are the positive solutions of the following algebraic equations :

$$rx(1-\frac{x}{Ax+K}) - \sigma_1 x + \sigma_2 y - \beta_1 xz = 0, \tag{5a}$$

$$sy(1-\frac{y}{L}) + \sigma_1 x - \sigma_2 y = 0\,, \tag{5b}$$

$$z = \frac{M}{a}(a + \beta_2 x). \tag{5c}$$

Solving the above system of algebraic equations, we get

$$Ax^5 + Bx^4 + Cx^3 + Dx^2 + Ex + F = 0 \tag{6}$$

Where

$$A = \left[\frac{s}{L\sigma_2{}^2}\left(\frac{\beta_1{}^2\beta_2{}^2M^2A^2}{K^2a^2}\right)\right]$$

$$B = \frac{s}{L\sigma_2{}^2}\left[\frac{2\beta_1\beta_2 MA}{K^2a}\{r - A(r-\sigma_1-\beta_1 M) + \frac{\beta_1\beta_2 MK}{a}\}\right]$$

$$C = \frac{s}{L\sigma_2{}^2}\left[\frac{r}{K} - \frac{(r-\sigma_1-\beta_1 M)A}{K} + \frac{\beta_1\beta_2 M}{a}\right]^2 - 2\left[\frac{\beta_1\beta_2 MA}{K^2a}(r-\sigma_1-\beta_1 M)\right]$$
$$-\frac{s-\sigma_2}{\sigma_2}\left[\frac{\beta_1\beta_2 MA^2}{K^2a}\right]$$

$$D = \frac{-2s}{L\sigma_2{}^2}\left[\frac{r}{K} - \frac{(r-\sigma_1-\beta_1 M)A}{K} + \frac{\beta_1\beta_2 M}{a}\right](r-\sigma_1-\beta_1 M$$
$$-\frac{s-\sigma_2}{\sigma_2}\left[\frac{rA}{K^2} - \frac{(r-\sigma_1-\beta_1 M)A^2}{K^2} + \frac{2\beta_1\beta_2 AM}{Ka}\right] - \left[\frac{\sigma_1 A^2}{K^2}\right]$$

$$E = \frac{s}{L\sigma_2^{\,2}}\left[(r-\sigma_1-\beta_1 M)^2 + \frac{2\sigma_1 A}{K}\right] - \frac{s-\sigma_2}{\sigma_2}\left[\frac{r}{K} - (r-\sigma_1-\beta_1 M)\frac{2A}{K} + \frac{\beta_1\beta_2 M}{a}\right] - \frac{2\sigma_1 A}{K}$$

$$F = \frac{s-\sigma_2}{\sigma_2}(r-\sigma_1-\beta_1 M) - \sigma_1$$

We note that the equation (6) has a real positive root x = x* if the following conditions are satisfied:

$$s\left[(r-\sigma_1-\beta_1 M)^2 + \frac{2\sigma_1 A}{K}\right] < L\sigma_2(s-\sigma_2) \left[\frac{r}{K} - (r-\sigma_1-\beta_1 M)\frac{2A}{K} + \frac{\beta_1\beta_2 M}{a}\right] + \frac{2\sigma_1 A}{K} \quad (7a)$$

$$(r-\sigma_1-\beta_1 M)(s-\sigma_2) < \sigma_1\sigma_2, \quad (7b)$$

$$(r-\sigma_1-\beta_1 M) > 0. \quad (7c)$$

Knowing the value of x*, the value of z* can be computed from equation (5c) and the value of y* can be computed from the equation given below

$$y^* = \frac{1}{\sigma_2}\left[\left(\frac{r}{Ax+K} + \frac{\beta_1\beta_2 M}{a}\right)x^{*2} - (r-\sigma_1-\beta_1 M)x^*\right] \quad (8)$$

For y* to be positive, we must have

$$\left(\frac{r}{Ax+K} + \frac{\beta_1\beta_2 M}{a}\right)x^* > (r-\sigma_1-\beta_1 M). \quad (9)$$

From the following lemma , we show that the model system (1) is biologically well behaved

Lemma .1

The set

$$\Omega = \{(x,y,z) : \omega(t) = x(t)+y(t)+z(t), 0 < \omega(t) \le \frac{\mu^*}{\eta^*}\}$$

Is a region of the attraction for all solutions initiating in the interior of the positive octant.

Where η^* is a constant such that

$$\mu^* = \frac{K}{4r}(r+\eta^*)^2 + \frac{L}{4s}(s+\eta^*)^2 + \frac{M}{4a}(a+\eta^*)^2$$

Proof . Let $\omega(t) = x(t)+y(t)+z(t)\ and\ \eta > 0$ be a constant. Then

$$\frac{dw}{dt} + \eta^* w = \frac{dx}{dt} + \frac{dy}{dt} + \frac{dz}{dx} + \eta^*(x+y+z)$$
$$= (r+\eta^*)x - \frac{rx^2}{Ax+K} + (s+\eta^*)y - \frac{sy^2}{L} + (a+\eta^*)z - \frac{az^2}{M} - (\beta_1-\beta_2)xz \quad (10)$$

Since β_1 is the depletion rate coefficient of prey due to its intake by the predator and β_2 is the growth rate coefficient of predator due to its interaction with their prey , and hence it is natural to assume that $\beta_1 \ge \beta_2$.

Then equation (10) can be written as

$$\frac{dw}{dt} + \eta^* w = (r+\eta^*)x - \frac{rx^2}{Ax+K} + (s+\eta^*)y - \frac{sy^2}{L} + (a+\eta^*)z - \frac{az^2}{M} \quad (11)$$

$$\frac{dw}{dt} + \eta^* w \le \frac{K}{4r}(r+\eta^*)^2 + \frac{L}{4s}(s+\eta^*)^2 + \frac{M}{4a}(a+\eta^*)^2 = \mu^* \quad (12)$$

$$\frac{dw}{dt} + \eta^* w = \mu^*$$

By using the differential inequality [5] ,we obtain

$$w\{x(t), y(t), z(t)\} \le \frac{\mu^*}{\eta^*}(1-e^{-\eta^* t}) + \{x(0), y(0), z(0)\}e^{-\eta^* t}$$

Taking limit when $t \to \infty$, we have, $0 < w(t) \le \frac{\mu^*}{\eta^*}$, proving the lemma.

4. STABILITY ANALYSIS

Study the local stability behavior of F^* ,we compute the variational matrices corresponding to each equilibrium. From the system (1) is

$$J = \begin{pmatrix} r - r(\frac{2xk+x^2A}{(Ax+k)^2}) - \sigma_1 - \beta_1 z & \sigma_2 & -\beta_1 x \\ \sigma_1 & s - \frac{2sy}{L} - \sigma_2 & 0 \\ \beta_2 x & 0 & a - \frac{2az}{M} + \beta_2 x \end{pmatrix} \quad (13)$$

The characteristic equation of the variational matrix (13) at $F_0\,(0,0,0)$

$$J_0=\begin{pmatrix} r-\sigma_1 & \sigma_2 & 0\\ \sigma_1 & s-\sigma_2 & 0\\ 0 & 0 & a\end{pmatrix}$$

1. F_0 is an unstable equilibrium point.

The characteristic equation of the variational matrix (13) at $F_1(0,0,M)$

$$J_1=\begin{pmatrix} r-\sigma_1-\beta_1 M & \sigma_2 & 0\\ \sigma_1 & s-\sigma_2 & 0\\ 0 & 0 & -a\end{pmatrix}$$

2. F_1 is a saddle point whose stable manifold is locally in the z- direction and unstable manifold locally in the xy- plane.

The characteristic equation of the variational matrix of (13) at $F_2(\tilde{x},\tilde{y},0)$

$$J_2=\begin{pmatrix} r-r(\frac{2xk+x^2A}{(Ax+k)^2})-\sigma_1 & \sigma_2 & -\beta_1 x\\ \sigma_1 & s-\frac{2sy}{L}-\sigma_2 & 0\\ \beta_2 x & 0 & a+\beta_2 x\end{pmatrix}$$

3. F_2 is also a saddle point whose stable manifold is locally in the xy-plane and unstable manifold locally in the z- direction.

In the following theorem, we show that F^* is locally asymptotically stable.

Theorem 1. The interior equilibrium F^* is locally asymptotically stable.

Proof. In order to prove this theorem. We first linearize model (1) by taking the following transformation.

$$x=\bar{x}+X,\quad y=\bar{y}+Y,\quad z=\bar{z}+Z \tag{14}$$

Now we considered the following positive definite function:

$$V(t)=\frac{1}{2}X^2+\frac{1}{2}c_1Y^2+\frac{1}{2}c_2Z^2 \tag{15}$$

Where c_1 and c_2 are positive constant to chosen suitably.

Now differentiating 'V' with respect to time t along the linear version of model (1) we get

$$\frac{dV}{dt}=-\left(\frac{r\bar{x}}{Ax+K}+\frac{\sigma_2\bar{y}}{\bar{x}}\right)X^2-c_1\left(\frac{s\bar{y}}{L}+\frac{\sigma_1\bar{x}}{\bar{y}}\right)Y^2$$
$$-c_2\left(\frac{a\bar{z}}{M}+\frac{\beta_2\bar{x}}{\bar{z}}\right)Z^2+XZ\left(c\,\beta_2+\beta_1 x_1\right)+\left(\ _2\ _2-\ _1\bar{}\ \right)$$

Choosing $c_2=\frac{\beta_1\bar{x}}{\beta_2}$ we note that V is negative definite if

$$\left(\sigma_2+c_1\sigma_1\right)^2<4c_1\left(\frac{s\bar{y}}{L}+\frac{\sigma_1\bar{x}}{\bar{y}}\right)\left(\frac{r\bar{x}}{Ax+K}+\frac{\sigma_2\bar{y}}{\bar{x}}\right)\left(-\frac{\beta_1\bar{x}}{\beta_2}(\frac{\beta_2\bar{z}}{\bar{z}}+\frac{a\bar{z}}{\beta_2})z^2\right) \tag{16}$$

It can be written as

$$\left(\sigma_2-c_1\sigma_1\right)^2+4c_1\sigma_1\sigma_2\le 4c_1\left(\frac{s\bar{y}}{L}+\frac{\sigma_1\bar{x}}{\bar{y}}\right)\left(\frac{r\bar{x}}{Ax+K}+\frac{\sigma_2\bar{y}}{\bar{x}}\right)\left(-\frac{\beta_1\bar{x}}{\beta_2}(\frac{\beta_2\bar{z}}{\bar{z}}+\frac{a\bar{z}}{\beta_2})z^2\right)$$

If we choose $c_1=\frac{\sigma_2}{\sigma_1}$, then above condition is satisfied and show that V is Liapunov function .

5. CONCLUSION

A nonlinear mathematical model is proposed and analyzed to see the effect of harvesting and prey reserve on prey-predator dynamics. Using stability theory of differential equations, we have obtained conditions for the existence of different equilibria and discussed their stabilities in local manner. Using, differential inequality, conditions have been obtained under which system persists and the model is globally asymptotically stable. It has been found that prey species has oscillatory behavior in the unreserved zone where as oscillatory behavior has not been observed for prey species in the reserved zone.

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